

## Free-surface supercritical splashless flows past a two-dimensional symmetrical rectilinear body

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**ABSTRACT.** – Two-dimensional free-surface potential flow past a ship in water of finite depth is considered. The ship is modelled as a symmetrical rectilinear body with inclined faces. Gravity is included in the dynamic boundary condition. It is assumed that the intersections of the free surface with the hull are stagnation points. Numerical solutions are obtained by a series truncation procedure when the flow is supercritical in the far field. It is found that there is a two-parameter family of splashless solutions. Some of the solutions can be viewed as perturbations of solitary waves. © Elsevier, Paris.

### 1. Introduction

We consider the steady two-dimensional irrotational flow of an inviscid incompressible fluid past a rectilinear symmetrical object lying on the free surface in water of finite depth (see *Fig. 1*). When the level of the bottom of the object is below the level of the free surface at infinity (like in *Figure 1*), the problem models a barge-like vessel moving at a constant velocity in a canal. When it is above, we refer to it as a surfing flow. Observations of real ships and the theoretical considerations described in the next paragraphs suggest that there is in general a splash at the bow of the vessel. In this paper we show that there are particular solutions without splashes. We restrict our attention to supercritical flows, *i.e.* flows for which the Froude number

$$(1) \quad F = U/\sqrt{gH}$$

is greater than one. Here  $U$  and  $H$  are the velocity and water-depth in the far field and  $g$  is the acceleration of gravity. We also assume that the flows are symmetric with respect to the  $y$ -axis.

The problem of free-surface flows past two-dimensional objects has been considered by many previous investigators. Analytical and numerical results have been proposed for different flow configurations. In water of infinite depth, Vanden-Broeck and Tuck (1977), Vanden-Broeck, Schwartz and Tuck (1978), and Vanden-Broeck (1980) showed that there are no continuous solutions for flows past a semi-infinite two-dimensional object with a flat bottom and a vertical front such that the flow separates at a stagnation point and approaches a uniform stream in the far field. The implication is that there is either a train of waves on the free surface or a splash. These two situations correspond respectively to near stern flows and near bow flows. The near stern flows were calculated in Vanden-Broeck and Tuck (1977), Vanden-Broeck (1980) and the near bow flow was constructed in Dias and Vanden-Broeck (1993). Furthermore Madurasinghe and Tuck (1986) constructed a near-bow flow without splash by replacing the assumption of a stagnation point with that of a smooth detachment.

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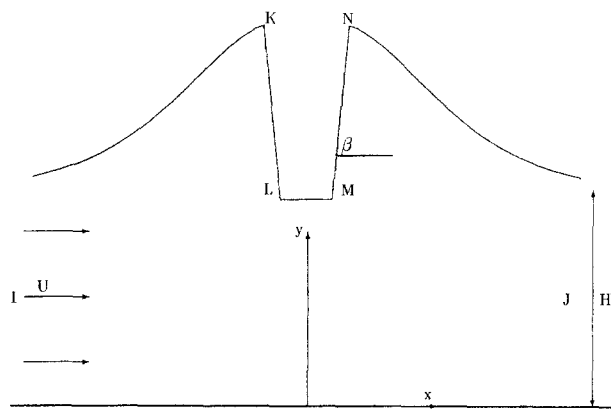


Fig. 1. – The flow past an object KLMN with stagnation points. The free surfaces are IK and NJ. The fluid has depth  $H$  and velocity  $U$  as  $|x| \rightarrow \infty$ . The profile shown is a computed solution for  $\beta = \frac{2\pi}{5}$ ,  $\delta_1 = \frac{1}{4}$ ,  $\delta_2 = \frac{1}{2}$  and  $F = 1.23$ . The ratio of the horizontal scale to the vertical scale is 3.5:1.

Vanden-Broeck (1989) generalized the flow configuration in Vanden-Broeck and Tuck (1977), Vanden-Broeck (1980) (*i.e.* the flow past a semi-infinite two-dimensional object with a flat bottom, a vertical front and a stagnation point) to water of finite depth. He showed that there are splashless supercritical flows. The corresponding problem with smooth detachment was considered by Hocking (1993). All the above results are for semi-infinite objects. Results for objects of finite length were obtained analytically by Craig and Sternberg (1991) and numerically by Asavanant and Vanden-Broeck (1994). These authors assume that the free surfaces separate smoothly from the object.

In the present paper we examine the flow past an object of finite length when there are stagnation points at the intersection of the free surface with the hull (see Fig. 1). The problem is solved by a series truncation method for arbitrary values of the width and the height of the object. The numerical procedure is similar to the one used by Vanden-Broeck and Keller (1989), Vanden-Broeck (1989) and Asavanant and Vanden-Broeck (1996). Our results reduce to those obtained by Vanden-Broeck (1989) as the length of the object tends to infinity. We show that for given inclinations of the faces, there is a two-parameter family of splashless solutions. Physically we expect that the flow configuration of Figure 1 depends on three parameters: the width of the ship, the draft (*i.e.* the distance between the bottom of the hull and the level of the free surface at infinity) and the Froude number. Furthermore we expect the members of this three-parameter family to have a splash at the bow. The findings of the present paper identifies among these members a sub-family (depending on two parameters) of splashless solutions. Splashless solutions are of particular interest, since an important concern in ship hydrodynamics is the reduction of the splash at the bow of a ship. The solutions presented here supplement those in Asavanant and Vanden-Broeck (1994), where the free surfaces were assumed to separate smoothly from the object.

In section 2 we formulate the problem for the flow configuration shown in Figure 1. The numerical procedure is described in section 3. In section 4 we discuss the numerical results. Concluding remarks are presented in section 5.

2. Formulation of the problem

We consider the flow configuration shown in Figure 1. The hull is assumed to have a flat bottom and two faces (inclined at an angle  $\beta$ ) from which the free surface separates at the stagnation points K and N. We introduce cartesian coordinates with the  $x$ -axis along the bottom, the  $y$ -axis directed vertically upwards and  $x = 0$  in the middle of the bottom of the hull. Gravity is acting in the negative  $y$ -direction. The coordinate

system is moving with the hull, so that the flow is steady. We assume that the flow is supercritical in the far field (*i.e.*  $F \geq 1$ ). Therefore there are no waves on the free surface and the flow approaches a uniform stream with constant velocity  $U$  and uniform depth  $H$  as  $|x| \rightarrow \infty$ .

It is convenient to define dimensionless variables by taking  $U$  as the unit velocity and  $H$  as the unit length. We introduce the velocity potential  $\phi(x, y)$  and the streamfunction  $\psi(x, y)$ . Next we define the complex potential  $f = \phi + i\psi$  and the complex velocity by  $\zeta = u - iv = df/dz$ . Here  $u$  and  $v$  are the velocity components in the  $x$  and  $y$  directions, and  $z = x + iy$ . Without loss of generality, we choose  $\phi = 0$  at the point in the middle of the bottom LM of the object and  $\psi = 1$  on the free surface IK, NJ and on the object KLMN. It follows that  $\psi = 0$  on the bottom. The flow domain in the  $f$ -plane is an infinite strip (see Fig. 2). We denote by  $\pm a$  and  $\pm b$  the values of the potential function at the corner points L and M and at the two separation points K and N respectively (see Fig. 2). On the free surface, the pressure is constant and the Bernoulli equation in dimensionless form yields

$$(2) \quad |\zeta|^2 + 2(y - 1)/F^2 = 1; \quad \text{on IK and NJ.}$$

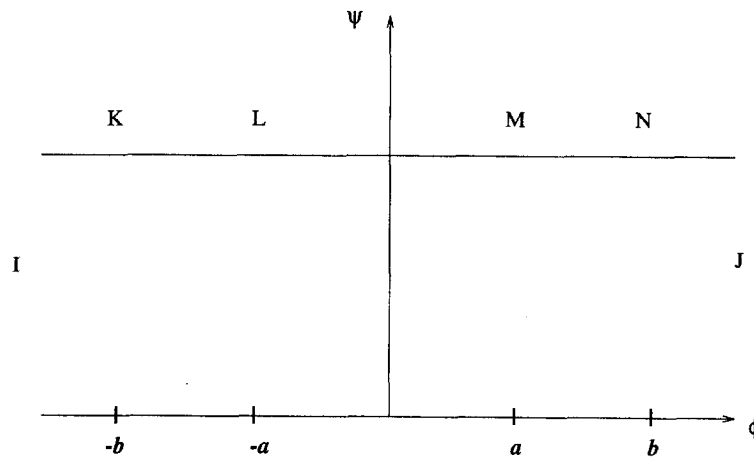


Fig. 2. – Flow configuration in the complex potential plane  $f = \phi + i\psi$ .

Here  $F$  is the Froude number defined by (1).

The kinematic condition on the bottom IJ, and on the object KL, LM, MN can be expressed as

$$(3) \quad \text{Im}\zeta = 0 \quad \text{on} \quad \psi = 0, \quad -\infty < \phi < \infty,$$

$$(4) \quad |\text{Re}\zeta/\text{Im}\zeta| = \cotan\beta \quad \text{on}, \quad \psi = 1 \quad -b < \phi < -a \quad \text{and} \quad a < \phi < b,$$

$$(5) \quad \text{Im}\zeta = 0 \quad \text{on} \quad \psi = 1, \quad -a < \phi < a.$$

As  $|\phi| \rightarrow \infty$ , the flow approaches a uniform stream with constant unit velocity. It can easily be shown by linearizing around a uniform stream that the approach is described by exponentially decaying terms, *i.e.*

$$(6) \quad \zeta \sim 1 + De^{\mp\pi\lambda f} \quad \text{as} \quad \phi \rightarrow \pm\infty.$$

Here  $D$  is a constant to be determined as part of the solution and  $\lambda$  is the smallest positive root of

(7) 
$$\pi \lambda F^2 - \tan \pi \lambda = 0.$$

At the stagnation points K and N, the flow is locally a flow inside a  $120^\circ$  angle when  $0 \leq \beta \leq \pi/3$ . When  $\pi/3 \leq \beta \leq \pi/2$ , the free surface is horizontal at the stagnation points (see Dagan and Tulin, 1972 for details). Therefore

(8) 
$$\zeta \sim (f \pm b - i)^{\theta/\pi} \quad \text{as } f \rightarrow \mp b + i$$

where

$$\begin{aligned} \theta &= \beta \quad \text{if } \pi/3 \leq \beta \leq \pi/2, \\ \theta &= \pi/3 \quad \text{if } 0 \leq \beta \leq \pi/3. \end{aligned}$$

At the corner point, the singularity is given by

(9) 
$$\zeta \sim (f \pm a - i)^{-\beta/\pi} \quad \text{as } f \rightarrow \mp a + i$$

The problem now becomes that of finding  $\zeta$  as an analytic function of  $f$  in the strip  $0 < \psi < 1$  satisfying the Eqs. (2)-(6), (8) and (9).

3. Numerical procedure

We solve the problem numerically by series truncation. This approach has been used by many previous authors (see for example Birkhoff and Carter (1957), and Lenau (1966), some early applications). Following Vanden-Broeck (1989) and Asavanant and Vanden-Broeck (1996), we map the flow domain from the complex  $f$ - plane onto the upper half of the unit circle in the complex  $t$ -plane. The transformation is given by

(10) 
$$f = (2/\pi) \log [(1 + t)/(1 - t)].$$

Its maps the bottom IJ onto the real diameter, and the free surface IK, NJ and the object KL, LM, MN onto the circumference (see Fig. 3). We use the notation  $t = re^{i\sigma}$  so that the free surface and the object are described by  $r = 1$  and  $0 < \sigma < \pi$ . The points  $t = e^{i\gamma_1}$  and  $t = -e^{-i\gamma_1}$  are the images of the stagnation points N and K. The points  $t = e^{i\gamma_2}$  and  $t = -e^{-i\gamma_2}$  are the images of the corner points M and L of the object. By using (10), we find that  $\gamma_1$  and  $\gamma_2$  are related to  $b$  and  $a$  by

(11) 
$$\gamma_1 = 2 \arctan[\exp(-\pi b/2)]$$

(12) 
$$\gamma_2 = 2 \arctan[\exp(-\pi a/2)].$$

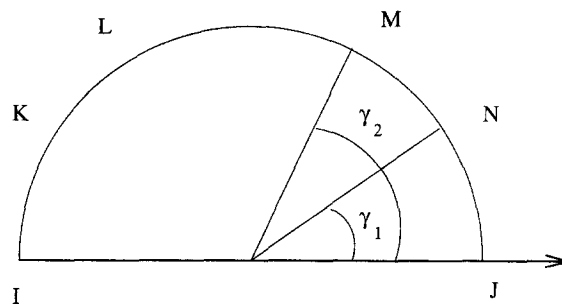


Fig. 3. – The image of the flow of Figure 1 in the complex  $t$ - plane is the upper half unit disk.

We now seek the complex velocity  $\zeta$  as a function of  $t$ . Taking into account the local behaviors of the flow in (6), (8), (9) and the symmetry of the flow about  $y = 0$ , we write the complex velocity as

$$(13) \quad \zeta = [((t^2 + 1)^2 - 4t^2 \cos^2 \gamma_1)/(4 - 4 \cos^2 \gamma_1)]^{\theta/\pi} [((t^2 + 1)^2 - 4t^2 \cos^2 \gamma_2)/(4 - 4 \cos^2 \gamma_2)]^{-\beta/\pi} e^{\Omega(t)}$$

where  $\Omega(t)$  has the expansion

$$(14) \quad \Omega(t) = A(1 - t^2)^{2\lambda} + \sum_{n=1}^{\infty} a_n(t^{2n} - 1).$$

The kinematic condition (3) on the bottom  $IJ$  implies the coefficients  $A$  and  $a_n$  are real. The representation (13) factors out the singular behaviors of the velocity at the corner points and the stagnation points. It can easily be verified that (13) satisfies (6), (8), and (9). Therefore we can expect the expansion in (14) to converge for  $|t| \leq 1$ . The unknown constant  $A$  and the coefficients  $a_n$  of the power series must be determined so that the dynamic boundary condition (2) on the free surface, and the kinematic conditions (4) and (5) on the object are satisfied. We first eliminate  $y$  from (2) by differentiating this equation with respect to  $\sigma$ . By using the identity

$$(15) \quad \partial x / \partial \phi + i \partial y / \partial \phi = 1 / \zeta.$$

we obtain

$$(16) \quad F^2[u(\sigma)u_{\sigma}(\sigma) + v(\sigma)v_{\sigma}(\sigma)] - 2/(\pi \sin \sigma)[v(\sigma)/(u^2(\sigma) + v^2(\sigma))] = 0.$$

We now solve the problem numerically by truncating the infinite series in (14) after  $N$  terms. There are  $N + 3$  unknowns  $\lambda$ ,  $A$ ,  $F$  and the coefficients  $a_n$  to be determined by collocation. Thus we introduce the  $N + 2$  mesh points

$$(17) \quad \sigma_1 = (\pi/[2(N + 2)])(I - 1/2), \quad I = 1, \dots, N + 2.$$

Here we take advantage of the symmetry of the problem by using mesh points only for  $0 \leq \sigma \leq \pi/2$ . For simplicity, we consider values of  $\gamma_1$  and  $\gamma_2$  in the form of

$$(18) \quad \begin{aligned} \gamma_1 &= \pi M_1/[2(N + 2)] \\ \gamma_2 &= \pi M_2/[2(N + 2)]. \end{aligned}$$

where  $M_1$  and  $M_2$  are integers smaller than  $N + 2$  and  $M_1 \leq M_2$ . We obtain  $N + 2$  equations by satisfying (16) at the mesh points  $I = 1, \dots, M_1$ , (4) at the mesh points  $I = M_1 + 1, \dots, M_2$ , and (5) at the mesh points

$I = M_2 + 1, \dots, N + 2$ . The last equation is provided by imposing the relation (7). For given values of  $\beta$ ,  $M_1$  and  $M_2$ , we solve this system of  $N + 3$  nonlinear algebraic equations with  $N + 3$  unknowns by Newton’s method. Once it is solved we obtain the shape of the free surface and the object by integrating numerically the relations

(19) 
$$dx/d\sigma = -2/(\pi \sin \sigma)[u(\sigma)/(u^2(\sigma) + v^2(\sigma))]$$

(20) 
$$dy/d\sigma = -2/(\pi \sin \sigma)[v(\sigma)/(u^2(\sigma) + v^2(\sigma))].$$

4. Discussion of the results

The numerical scheme described in the previous section was used to compute solutions for various inclinations of the faces and various values of  $\gamma_1$  and  $\gamma_2$ . Using (18), we specify  $\gamma_1$  and  $\gamma_2$  by fixing

$$\delta_1 = M_1/(N + 2) \quad \text{and} \quad \delta_2 = M_2/(N + 2).$$

The coefficients  $a_n$  were found to decrease rapidly. For example,  $|a_{10}/a_1| \approx 0.29 \times 10^{-1}$ ,  $|a_{40}/a_1| \approx 0.36 \times 10^{-2}$ ,  $|a_{200}/a_1| \approx 0.24 \times 10^{-3}$ ,  $|a_{370}/a_1| \approx 0.15 \times 10^{-6}$  for  $\beta = \pi/2$ ,  $\delta_1 = 1/2$  and  $\delta_2 = 7/10$ . Most of the calculations were performed with 400 coefficients. In all the calculations presented we checked that the results are independent of  $N$  within graphical accuracy.

We first present results for  $\beta = \pi/2$  (*i.e.* for vertical faces).

Typical profiles are shown in Figures 4 and 5. The length and the height of the object are shown in Figures 6 and 7 as functions of  $\delta_1$  and  $\delta_2$ .

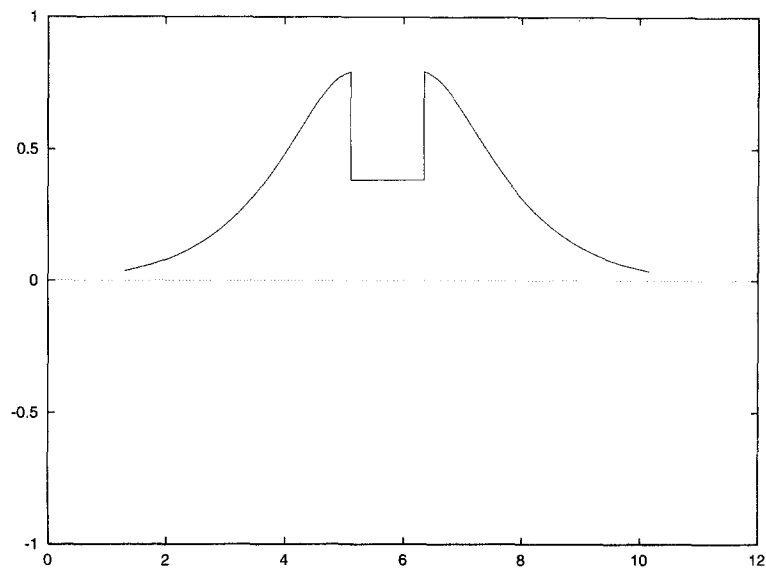


Fig. 4. – Computed profile for  $\beta = \frac{\pi}{2}$ ,  $\delta_1 = \frac{2}{5}$ ,  $\delta_2 = \frac{1}{2}$  and  $F = 1.27$ . The broken line indicates the level of the free surface as  $|x| \rightarrow \infty$ .

In Figure 5, the bottom of the object is below the level of the free surface at infinity. However the bottom of the object is above this level in Figure 4. Therefore Figure 4 does not model a ship. Following Vanden-Broeck and Keller (1989), we refer to this flow as a “surfing flow”. As  $\delta_1 \rightarrow 1$  and  $\delta_2 \rightarrow 1$ , the height MN, KL and the bottom width LM of the object reduce to zero and we recover the case of the steepest solitary wave.

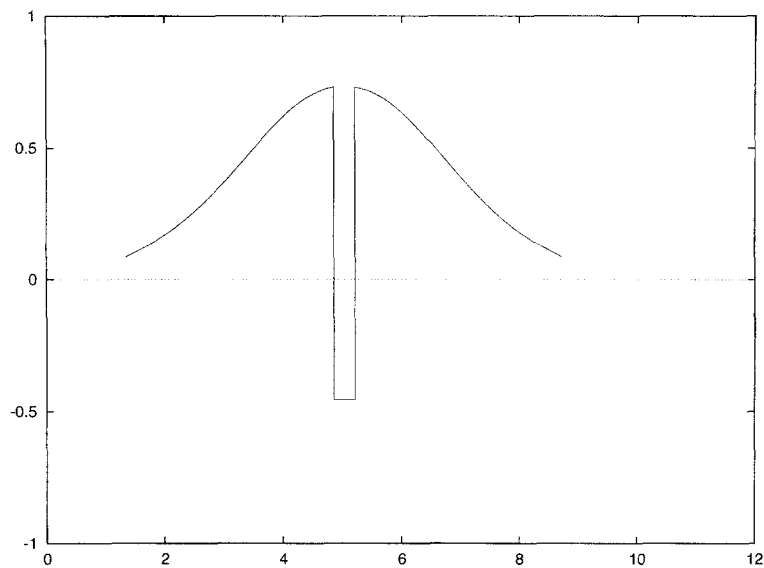


Fig. 5. – Computed profile for  $\beta = \frac{\pi}{2}$ ,  $\delta_1 = \frac{3}{20}$ ,  $\delta_2 = \frac{1}{2}$  and  $F = 1.23$ .

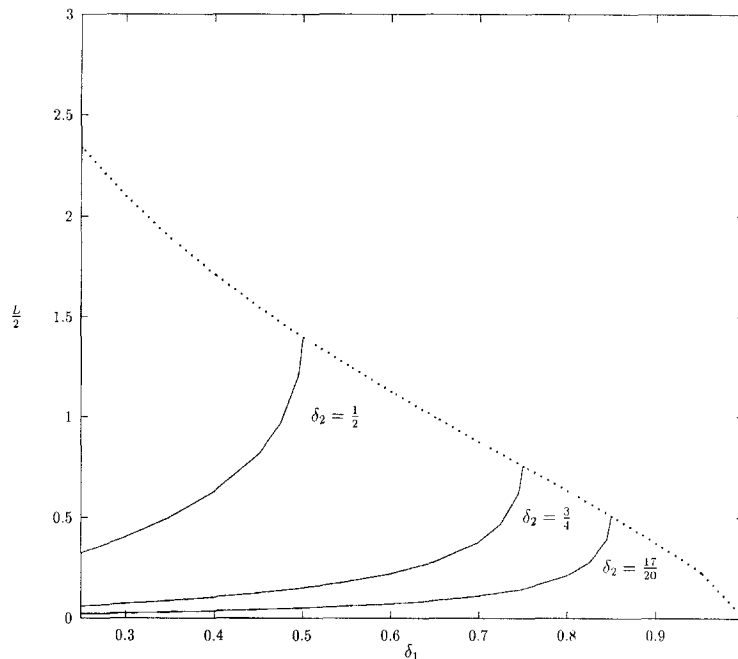


Fig. 6. – Values of  $\frac{L}{2}$  versus  $\delta_1$  for  $\beta = \frac{\pi}{2}$  and various values of  $\delta_2$ . Here  $L$  is the width of the bottom LM of the object (see Fig. 1). The broken line corresponds to the solution (22).

It is found that the limiting configuration with sharp crest and a  $120^\circ$  angle is obtained at  $F = 1.29$  (see Fig. 8). The value of this critical Froude number is found to be in good agreement with the one obtained by Asavanant and Vanden-Broeck (1994) and Hunter and Vanden-Broeck (1983). This constitutes a useful check on the accuracy of the numerical procedure.

In Figure 9, we present the values of the Froude number versus  $\delta_1$  for various values of  $\delta_2$ . These results show that there is a two-parameter family of splashless solutions.

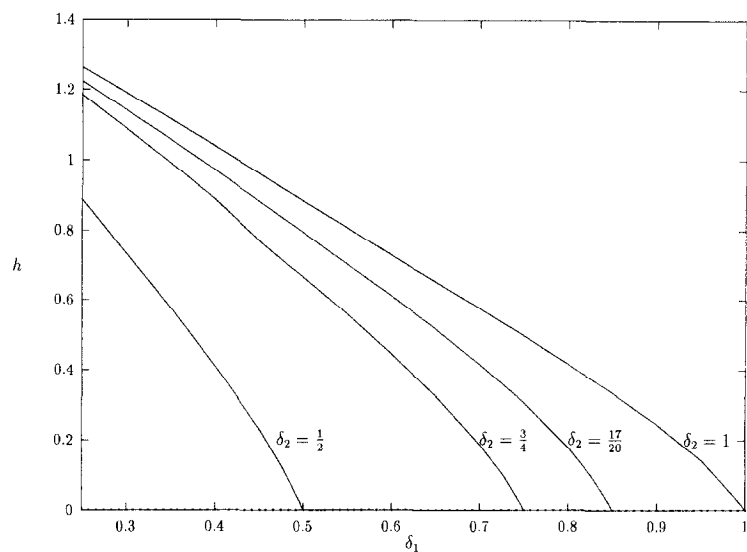


Fig. 7. – Values of  $h$  versus  $\delta_1$  for  $\beta = \frac{\pi}{2}$  and various values of  $\delta_2$ . Here  $h$  is the dimensionless height KL of the object.

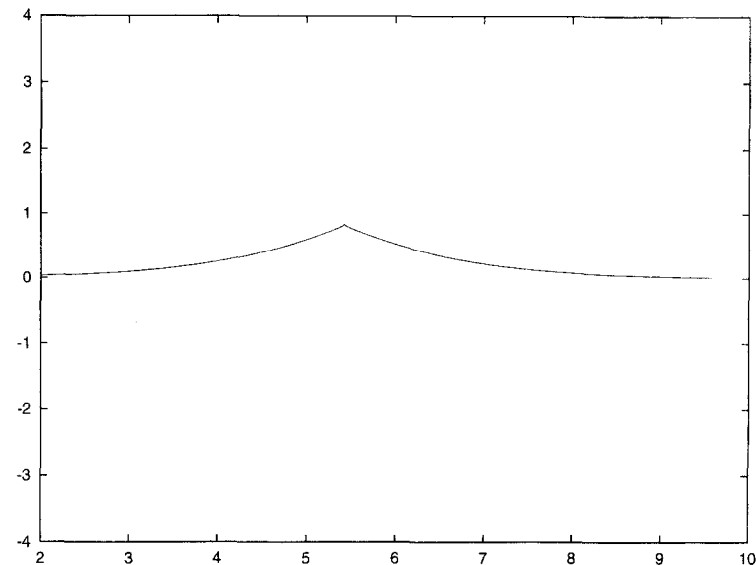


Fig. 8. – Highest solitary wave with  $F = 1.29$  when  $\delta_1, \delta_2 \rightarrow 1$ .

As  $\delta_1 \rightarrow \delta_2$ , the height MN, KL of the object approaches zero and the problem reduces to a flow past a flat plate considered by Vanden-Broeck and Keller (1989). It is a configuration with two stagnation points and  $120^\circ$  angles at the end of the plate. To compute accurately these flows, we note that (8) should be replaced by a singular behavior corresponding to a flow inside a  $120^\circ$ , *i.e.*

(21) 
$$\zeta \sim K(f \pm b - i)^{1/3} \quad \text{as } f \rightarrow \mp b + i.$$

The complex velocity  $\zeta$  is then expanded as

(22) 
$$\zeta = [((t^2 + 1)^2 - 4t^2 \cos^2 \gamma)/(4 - 4 \cos^2 \gamma)]^{1/3} \exp[A(1 - t^2)^{2\lambda} + \sum_{n=1}^\infty a_n(t^{2n} - 1)].$$



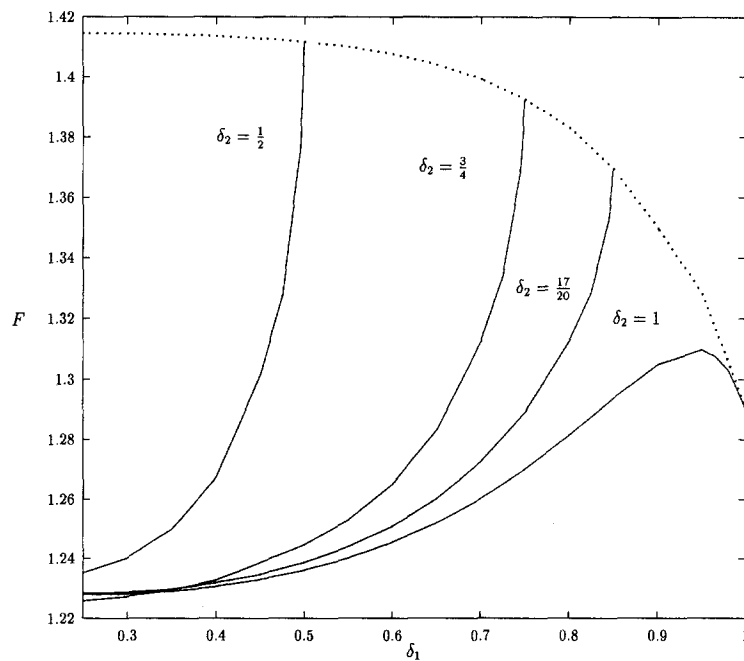


Fig. 9. – Relationship between the Froude number  $F$  and  $\delta_1$  for  $\beta = \frac{\pi}{2}$  and various values of  $\delta_2$ . The broken line corresponds to the solution (22).

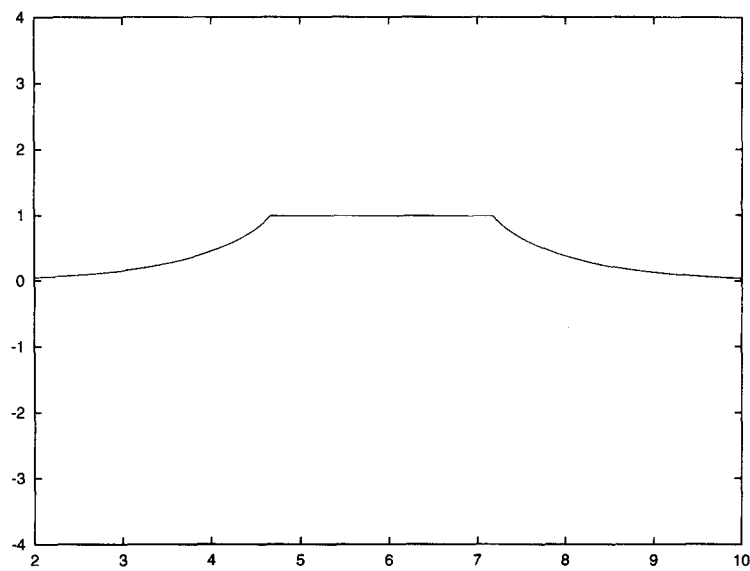


Fig. 10. – Computed flow profile of flow past a finite flat plate with  $120^\circ$  angle corners at the separation points for  $\delta_1 = 0.55$ ,  $\delta_2 = 1$  and  $F = 1.41$ .

Here  $\gamma = \gamma_1 = \gamma_2$ . A typical computed profile is shown in Figure 10 and the values of  $F$  versus  $\delta_1$  are shown in Figure 9 (broken line). The particular value  $\gamma = 0$  corresponds to a semi-infinite flat plate. It can then be shown analytically that  $F = \sqrt{2}$  (see Vanden-Broeck and Keller, 1989). The agreement between our numerical value of  $F$  for  $\delta_1 = 0$  and this value is another check on the numerical scheme.

We now define the amplitude parameter as

(23) 
$$\alpha = W/H,$$

where  $W$  is the distance from the bottom IJ to the bottom LM of the object. Thus the bottom of the object lies above the undisturbed free surface level when  $\alpha - 1 > 0$  and below when  $\alpha - 1 < 0$ . Numerical values of  $\alpha - 1$  versus  $\delta_1$  are presented in Figure 11 for various values of  $\delta_2$ .

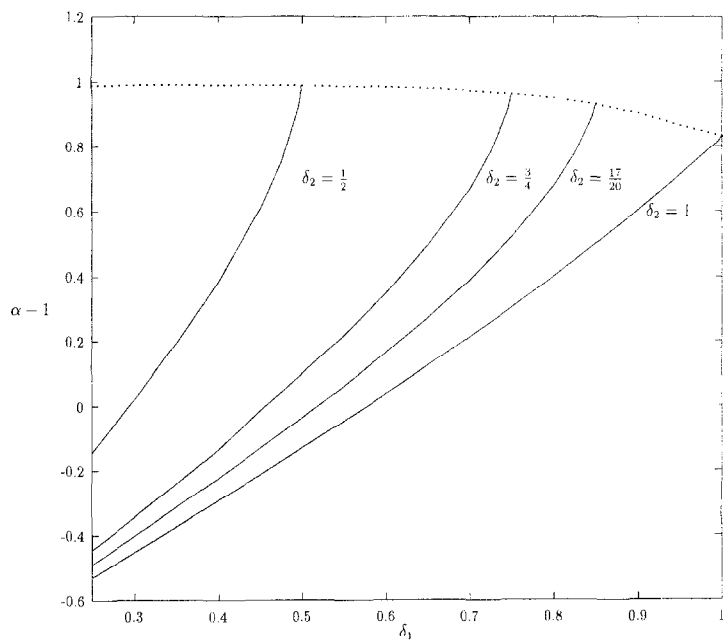


Fig. 11. – Values of dimensionless height above or below the level at infinity  $\alpha - 1$  versus  $\delta_1$  for various values of  $\delta_2$ . The broken curve corresponds to the solution (22).

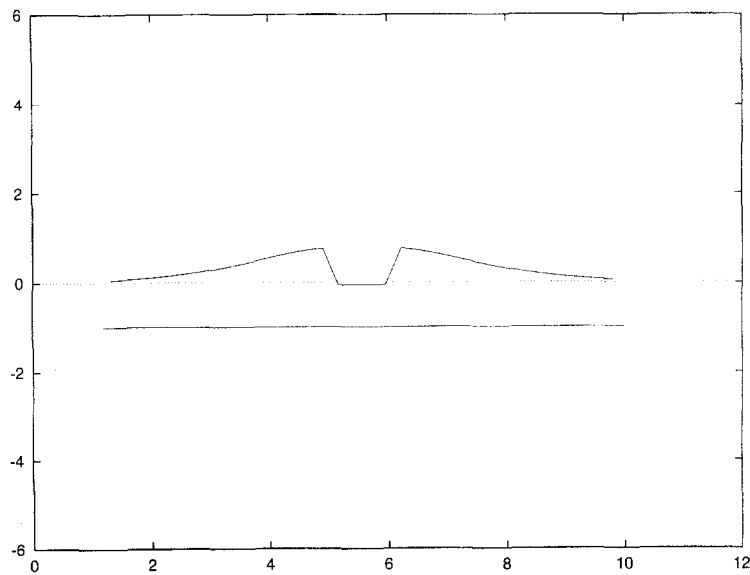


Fig. 12. – Computed profile for  $\delta_1 = \frac{1}{4}$ ;  $\delta_2 = \frac{1}{2}$ ;  $\beta = \frac{2\pi}{5}$  and  $F = 1.25$ .

We conclude by presenting two typical profiles for two values of face inclinations different than  $\pi/2$  in Figures 12 and 13.

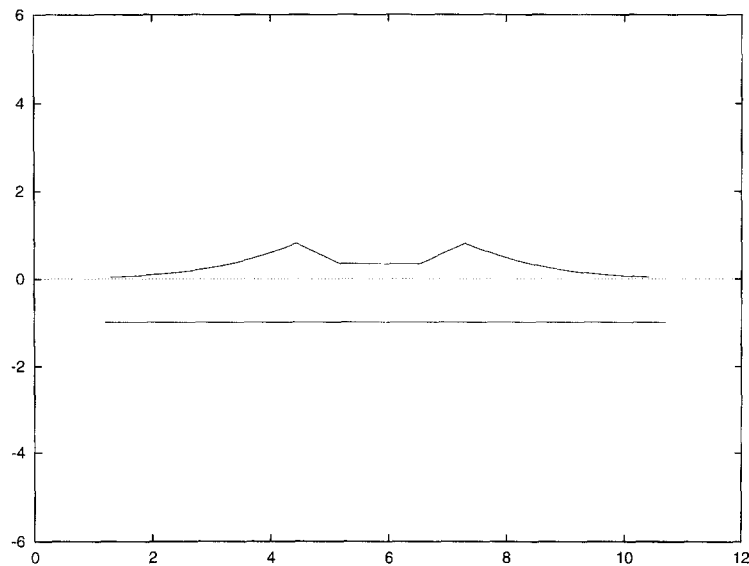


Fig. 13. – Computed profile for  $\delta_1 = \frac{1}{4}$ ;  $\delta_2 = \frac{1}{2}$   $\beta = \frac{\pi}{6}$  and  $F = 1.28$ .

## 5. Conclusions

We have presented numerical solutions for the free surface flow past a two-dimensional ship in water of finite depth. We have assumed that the free surface attaches to the hull at stagnation points. The results supplement previous studies (*see for example Asavanant and Vanden-Broeck, 1994*) in which smooth attachment is assumed. We have shown numerically that there is a two-parameter family of splashless solutions.

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